J421. Proposed by Adrian Andreescu, Dallas, USA

Let a and b be positive real numbers. Prove that

$$\frac{6ab - b^2}{8a^2 + b^2} < \sqrt{\frac{a}{b}}.$$

Solution by Arkady Alt, San Jose, California, USA.

Since $\frac{6ab - b^2}{8a^2 + b^2} < \sqrt{\frac{a}{b}} \iff \frac{6 \cdot \frac{a}{b} - 1}{8\left(\frac{a}{b}\right)^2 + 1} < \sqrt{\frac{a}{b}}$ then denoting $t := \sqrt{\frac{a}{b}} > 0$ we obtain

$\frac{6t^2 - 1}{8t^4 + 1} < t \iff 8t^5 + t + 1 > 6t^2$, where latter inequality holds because by AM-GM Inequality

$8t^5 + t + 1 > 3\sqrt[3]{8t^5 \cdot t \cdot 1} = 3\sqrt[3]{8t^6} = 6t^2$ (here is no equality sign because condition of equality in AM-GM Inequality ($8t^5 = t = 1$) is not fulfilled)